New special curves and their spherical indicatrices

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Abstract

In this paper, we define a new special curve in Euclidean 3-space which we call $k-$*slant helix* and introduce some characterizations for this curve. This notation is generalization of a general helix and slant helix. Furthermore, we have given some necessary and sufficient conditions for the $k-$slant helix.

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1 Introduction

From the view of differential geometry, a *straight line* is a geometric curve with the curvature $\kappa(s)=0$. A *plane curve* is a family of geometric curves with torsion $\tau(s)=0$. Helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [4]. The helix may be called a *circular helix* or W*-curve* [9]. It is known that straight line ($\kappa(s)=0$) and circle ($\kappa(s)=a$, $\tau(s)=0$) are degenerate-helices examples [12]. In fact, circular helix is the simplest three-dimensional spirals [6].

A curve of constant slope or *general helix* in Euclidean 3-space $\mathbf{E}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 (see [19] for details) says that: *A necessary and sufficient condition that a curve be a general helix is that the function*

$$f=\frac{\tau}{\kappa}$$

is constant along the curve, where κ and τ denote the curvature and the torsion, respectively. General helices or *inclined curves* are well known curves in classical differential

geometry of space curves and we refer to the reader for recent works on this type of curves [1, 2, 7, 15, 20].

In 2004, Izumiya and Takeuchi [10] have introduced the concept of *slant helix* by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the *geodesic curvature* of the principal image of the principal normal indicatrix

$$\sigma = \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'$$

is a constant function. Kula and Yayli [13] have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helices. Recently, Kula et al. [14] investigated the relation between a general helix and a slant helix. Moreover, they obtained some differential equations which are characterizations for a space curve to be a slant helix.

A family of curves with constant curvature but non-constant torsion is called Salkowski curves and a family of curves with constant torsion but non-constant curvature is called anti-Salkowski curves [17]. Monterde [16] studied some characterizations of these curves and he proved that the principal normal vector makes a constant angle with fixed straight line. So that: Salkowski and anti-Salkowski curves are the important examples of slant helices.

A unit speed curve of *constant precession* in Euclidean 3-space $\mathbf{E}^3$ is defined by the property that its (Frenet) Darboux vector

$$W = \tau\,\mathbf{T} + \kappa\,\mathbf{B}$$

revolves about a fixed line in space with constant angle and constant speed. A curve of constant precession is characterized by having

$$\kappa = \frac{\mu}{m}\sin[\mu\, s], \qquad \tau = \frac{\mu}{m}\cos[\mu\, s]$$

or

$$\kappa = \frac{\mu}{m}\cos[\mu\, s], \qquad \tau = \frac{\mu}{m}\sin[\mu\, s]$$

where μ and m are constants. This curve lies on a circular one-sheeted hyperboloid

$$x^2 + y^2 - m^2\, z^2 = 4m^2$$

The curve of constant precession is closed if and only if $n = \frac{m}{\sqrt{1+m^2}}$ is rational [18]. Kula and Yayli [13] proved that the geodesic curvature of the spherical image of the principal normal indicatrix of a curve of constant precession is a constant function equals $-m$. So, one can say that: the curves of constant precessions are the important examples of slant helices.

In this work, we define a new curve and we call it a $k-$*slant helix* and we introduce some characterizations of this curve. Furthermore, we have given some necessary and sufficient conditions for the $k-$slant helix. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling as well as other applications of interest.

2 Preliminaries

In Euclidean space $\mathbf{E}^3$, it is well known that each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are respectively, the tangent, the principal normal and the binormal vector fields [8].

We consider the usual metric in Euclidean 3-space $\mathbf{E}^3$, that is,

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\psi : I \subset \mathbb{R} \to \mathbf{E}^3$, $\psi = \psi(s)$, be an arbitrary curve in $\mathbf{E}^3$. The curve ψ is said to be of unit speed (or parameterized by the arc-length) if $\langle \psi'(s), \psi'(s)\rangle = 1$ for any $s \in I$. In particular, if $\psi(s) \neq 0$ for any s, then it is possible to re-parameterize ψ, that is, $\alpha = \psi(\phi(s))$ so that α is parameterized by the arc-length. Thus, we will assume throughout this work that ψ is a unit speed curve.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the moving frame along ψ, where the vectors $\mathbf{T}, \mathbf{N}$ and $\mathbf{B}$ are mutually orthogonal vectors satisfying $\langle \mathbf{T}, \mathbf{T}\rangle = \langle \mathbf{N}, \mathbf{N}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 1$. The Frenet equations for ψ are given by ([19, 20])

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}. \tag{1}$$

If $\tau(s) = 0$ for all $s \in I$, then $\mathbf{B}(s)$ is a constant vector V and the curve ψ lies in a 2-dimensional affine subspace orthogonal to V, which is isometric to the Euclidean 2-space $\mathbf{E}^2$.

3 New representation of spherical indicatrices

In this section we introduce a *new representation* of spherical indicatrices of the regular curves in Euclidean 3-space $\mathbf{E}^3$ by the following:

Definition 3.1. *Let ψ be a unit speed regular curve in Euclidean 3-space with Frenet vectors $\boldsymbol{T}$, $\boldsymbol{N}$ and $\boldsymbol{B}$. The unit tangent vectors along the curve $\psi(s)$ generate a curve $\psi_{\mathbf{t}} = \boldsymbol{T}$ on the sphere of radius* 1 *about the origin. The curve $\psi_{\mathbf{t}}$ is called the spherical indicatrix*

of $\boldsymbol{T}$ or more commonly, $\psi_{\mathbf{t}}$ is called tangent indicatrix of the curve ψ. If $\psi=\psi(s)$ is a natural representations of the curve ψ, then $\psi_{\mathbf{t}}(s)=\boldsymbol{T}(s)$ will be a representation of $\psi_{\mathbf{t}}$. Similarly, one can consider the principal normal indicatrix $\psi_{\mathbf{n}}=\boldsymbol{N}(s)$ and binormal indicatrix $\psi_{\mathbf{b}}=\boldsymbol{B}(s)$.

Lemma 3.2. *If the Frenet frame of the tangent indicatrix $\psi_{\mathbf{t}}=\boldsymbol{T}$ of a space curve ψ is $\{\boldsymbol{T}_{\mathbf{t}},\boldsymbol{N}_{\mathbf{t}},\boldsymbol{B}_{\mathbf{t}}\}$, then we have Frenet formula:*

$$\left[\begin{array}{c}\boldsymbol{T}_{\mathbf{t}}^{'}(s_{\mathbf{t}})\\ \boldsymbol{N}_{\mathbf{t}}^{'}(s_{\mathbf{t}})\\ \boldsymbol{B}_{\mathbf{t}}^{'}(s_{\mathbf{t}})\end{array}\right]=\left[\begin{array}{ccc}0&\kappa_{\mathbf{t}}&0\\ -\kappa_{\mathbf{t}}&0&\tau_{\mathbf{t}}\\ 0&-\tau_{\mathbf{t}}&0\end{array}\right]\left[\begin{array}{c}\boldsymbol{T}_{\mathbf{t}}(s_{\mathbf{t}})\\ \boldsymbol{N}_{\mathbf{t}}(s_{\mathbf{t}})\\ \boldsymbol{B}_{\mathbf{t}}(s_{\mathbf{t}})\end{array}\right],\tag{2}$$

where

$$\boldsymbol{T}_{\mathbf{t}}=\boldsymbol{N},\quad \boldsymbol{N}_{\mathbf{t}}=\frac{-\boldsymbol{T}+f\,\boldsymbol{B}}{\sqrt{1+f^2}},\quad \boldsymbol{B}_{\mathbf{t}}=\frac{f\,\boldsymbol{T}+\boldsymbol{B}}{\sqrt{1+f^2}},\tag{3}$$

and

$$s_{\mathbf{t}}=\int\kappa(s)ds,\quad \kappa_{\mathbf{t}}=\sqrt{1+f^2},\quad \tau_{\mathbf{t}}=\sigma\sqrt{1+f^2},\tag{4}$$

where

$$f=\frac{\tau(s)}{\kappa(s)}\tag{5}$$

and

$$\sigma=\frac{f'(s)}{\kappa(s)\Big(1+f^2(s)\Big)^{3/2}}\tag{6}$$

is the geodesic curvature of the principal image of the principal normal indicatrix of the curve ψ, $s_{\mathbf{t}}$ is natural representation of the tangent indicatrix of the curve ψ and equal the total curvature of the curve ψ and $\kappa_{\mathbf{t}}$ and $\tau_{\mathbf{t}}$ are the curvature and torsion of $\psi_{\mathbf{t}}$.

Therefore we can see that:

$$\frac{\tau_{\mathbf{t}}}{\kappa_{\mathbf{t}}}=\sigma.\tag{7}$$

Lemma 3.3. *If the Frenet frame of the principal normal indicatrix $\psi_{\mathbf{n}}=\boldsymbol{N}$ of a space curve ψ is $\{\boldsymbol{T}_{\mathbf{n}},\boldsymbol{N}_{\mathbf{n}},\boldsymbol{B}_{\mathbf{n}}\}$, then we have Frenet formula:*

$$\left[\begin{array}{c}\boldsymbol{T}_{\mathbf{n}}^{'}(s_{\mathbf{n}})\\ \boldsymbol{N}_{\mathbf{n}}^{'}(s_{\mathbf{n}})\\ \boldsymbol{B}_{\mathbf{n}}^{'}(s_{\mathbf{n}})\end{array}\right]=\left[\begin{array}{ccc}0&\kappa_{\mathbf{n}}&0\\ -\kappa_{\mathbf{n}}&0&\tau_{\mathbf{n}}\\ 0&-\tau_{\mathbf{n}}&0\end{array}\right]\left[\begin{array}{c}\boldsymbol{T}_{\mathbf{n}}(s_{\mathbf{n}})\\ \boldsymbol{N}_{\mathbf{n}}(s_{\mathbf{n}})\\ \boldsymbol{B}_{\mathbf{n}}(s_{\mathbf{n}})\end{array}\right],\tag{8}$$

where

$$\begin{cases}\boldsymbol{T}_{\mathbf{n}}=\frac{-\boldsymbol{T}+f\,\boldsymbol{B}}{\sqrt{1+f^2}},\\ \boldsymbol{N}_{\mathbf{n}}=\frac{\sigma}{\sqrt{1+\sigma^2}}\Big[\frac{f\,\boldsymbol{T}+\boldsymbol{B}}{\sqrt{1+f^2}}-\frac{\boldsymbol{N}}{\sigma}\Big],\\ \boldsymbol{B}_{\mathbf{n}}=\frac{1}{\sqrt{1+\sigma^2}}\Big[\frac{f\,\boldsymbol{T}+\boldsymbol{B}}{\sqrt{1+f^2}}+\sigma\,\boldsymbol{N}\Big],\end{cases}\tag{9}$$

and

$$s_{\mathbf{n}} = \int \kappa(s)\sqrt{1+f^2(s)}\,ds, \quad \kappa_{\mathbf{n}} = \sqrt{1+\sigma^2}, \quad \tau_{\mathbf{n}} = \Gamma\sqrt{1+\sigma^2}, \tag{10}$$

where

$$\Gamma = \frac{\sigma'(s)}{\kappa(s)\sqrt{1+f^2(s)}\Big(1+\sigma^2(s)\Big)^{3/2}}, \tag{11}$$

$s_{\mathbf{n}}$ is natural representation of the principal normal indicatrix of the curve ψ and $\kappa_{\mathbf{n}}$ and $\tau_{\mathbf{n}}$ are the curvature and torsion of $\psi_{\mathbf{n}}$.

Therefore we have:

$$\frac{\tau_{\mathbf{n}}}{\kappa_{\mathbf{n}}} = \Gamma. \tag{12}$$

Lemma 3.4. *If the Frenet frame of the binormal indicatrix $\psi_{\mathbf{b}} = \boldsymbol{B}$ of a space curve ψ is $\{\boldsymbol{T}_{\mathbf{b}}, \boldsymbol{N}_{\mathbf{b}}, \boldsymbol{B}_{\mathbf{b}}\}$, then we have Frenet formula:*

$$\begin{bmatrix} \boldsymbol{T}_{\mathbf{b}}'(s_{\mathbf{b}}) \\ \boldsymbol{N}_{\mathbf{b}}'(s_{\mathbf{b}}) \\ \boldsymbol{B}_{\mathbf{b}}'(s_{\mathbf{b}}) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_{\mathbf{b}} & 0 \\ -\kappa_{\mathbf{b}} & 0 & \tau_{\mathbf{b}} \\ 0 & -\tau_{\mathbf{b}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_{\mathbf{b}}(s_{\mathbf{b}}) \\ \boldsymbol{N}_{\mathbf{b}}(s_{\mathbf{b}}) \\ \boldsymbol{B}_{\mathbf{b}}(s_{\mathbf{b}}) \end{bmatrix}, \tag{13}$$

where

$$\boldsymbol{T}_{\mathbf{b}} = -\boldsymbol{N}, \quad \boldsymbol{N}_{\mathbf{b}} = \frac{\boldsymbol{T} - f\,\boldsymbol{B}}{\sqrt{1+f^2}}, \quad \boldsymbol{B}_{\mathbf{b}} = \frac{f\,\boldsymbol{T} + \boldsymbol{B}}{\sqrt{1+f^2}}, \tag{14}$$

and

$$s_{\mathbf{b}} = \int \tau(s)ds, \quad \kappa_{\mathbf{b}} = \frac{\sqrt{1+f^2}}{f}, \quad \tau_{\mathbf{b}} = -\frac{\sigma\sqrt{1+f^2}}{f}, \tag{15}$$

where $s_{\mathbf{b}}$ is natural representation of the binormal indicatrix of the curve ψ and equal the total torsion of the curve ψ and $\kappa_{\mathbf{b}}$ and $\tau_{\mathbf{b}}$ are the curvature and torsion of $\psi_{\mathbf{b}}$.

Therefore we obtain:

$$\frac{\tau_{\mathbf{b}}}{\kappa_{\mathbf{b}}} = -\sigma. \tag{16}$$

4 k-slant helix and its characterizations

In this section we generalize the concept of the general helix and a slant helix by a new curve which we call it k-slant helix.

Definition 4.1. *Let $\psi = \psi(s)$ a natural representation of a unit speed regular curve in Euclidean 3-space with Frenet apparatus $\{\kappa, \tau, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$. A curve ψ is called a k-slant helix if the unit vector*

$$\psi_{\kappa+1} = \frac{\psi_k'(s)}{\|\psi_k'(s)\|} \tag{17}$$

makes a constant angle with a fixed direction, where $\psi_0 = \psi(s)$ *and* $\psi_1 = \frac{\psi_0'(s)}{\|\psi_0'(s)\|}$.

From the above definition we can see that:

(1): The 0-*slant helix* is the curve whose the unit vector

$$\psi_1 = \frac{\psi_0'(s)}{\|\psi_0'(s)\|} = \frac{\psi'(s)}{\|\psi'(s)\|} = \mathbf{T}(s), \tag{18}$$

(which is the tangent vector of the curve ψ) makes a constant angle with a fixed direction. So that the 0-slant helix is the general helix.

By using the Frenet frame (1), it is easy to prove the following two well-known lemmas:

Lemma 4.2. *Let* $\psi : I \to \boldsymbol{E}^3$ *be a curve that is parameterized by arclength with intrinsic equations* $\kappa(s) \neq 0$ *and* $\tau(s) \neq 0$. *The curve* ψ *is a* 0*-slant helix or general helix (the vector* ψ_1 *makes a constant angle,* ϕ*, with a fixed straight line in the space) if and only if the function* $f(s) = \frac{\tau}{\kappa} = \cot[\phi]$.

Lemma 4.3. *Let* $\psi : I \to \boldsymbol{E}^3$ *be a curve that is parameterized by arclength with intrinsic equations* $\kappa(s) \neq 0$ *and* $\tau(s) \neq 0$. *The curve* ψ *is a* 0*-slant helix or general helix if and only the binormal vector* $\boldsymbol{B}$ *makes a constant angle with fixed direction.*

(2): The 1-*slant helix* is the curve whose the unit vector

$$\psi_2 = \frac{\psi_1'(s)}{\|\psi_1'(s)\|} = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{N}(s), \tag{19}$$

(which is the principal normal vector of the curve ψ) makes a constant angle with a fixed direction. So that the 1-slant helix is the slant helix.

If we using the Frenet frame (2) of the tangent indicatrix of the the curve ψ, it is easy to prove the following two lemmas. The first lemma is introduced in [3, 5, 10, 13, 14]. Here, we state this lemma and introduce *new representation and its simple proof* using spherical tangent indicatrix of the curve. The second lemma is a new.

Lemma 4.4. *Let* $\psi : I \to \boldsymbol{E}^3$ *be a curve that is parameterized by arclength with intrinsic equations* $\kappa(s) \neq 0$ *and* $\tau(s) \neq 0$. *The curve* ψ *is a* 1*-slant helix or slant helix (the vector* ψ_2 *makes a constant angle,* ϕ*, with a fixed straight line in the space) if and only if the function* $\sigma(s) = \frac{\tau_{\mathbf{t}}}{\kappa_{\mathbf{t}}} = \cot[\phi]$.

Proof: ($\Rightarrow$) Let $\mathbf{d}$ be the unitary fixed vector makes a constant angle, ϕ, with the vector $\psi_2 = \mathbf{N} = \mathbf{T_t}$. Therefore

$$\langle \mathbf{T_t}, \mathbf{d} \rangle = \cos[\phi]. \tag{20}$$

Differentiating the equation (20) with respect to the variable $s_{\mathbf{t}}$ and using Frenet equations (2), we get

$$\kappa_{\mathbf{t}} \langle \mathbf{N_t}, \mathbf{d} \rangle = 0. \tag{21}$$

Because $\kappa_{\mathbf{t}} = \sqrt{1+f^2} \neq 0$, then we have

$$\langle \mathbf{N_t}, \mathbf{d} \rangle = 0. \tag{22}$$

From the above equation, the vector $\mathbf{d}$ is perpendicular to the vector $\mathbf{N_t}$ and so that the vector $\mathbf{d}$ lies in the space consists with the vectors $\mathbf{T_t}$ and $\mathbf{B_t}$. Therefore the vector $\mathbf{d}$ makes a constant angles with the two vectors $\mathbf{T_t}$ and $\mathbf{B_t}$. Hence, the vector $\mathbf{d}$ can be written as the following form:

$$\mathbf{d} = \cos[\phi]\mathbf{T_t} + \sin[\phi]\mathbf{B_t}. \tag{23}$$

If we differentiate equation (23), we have

$$0 = (\cos[\phi]\kappa_{\mathbf{t}} - \sin[\phi]\tau_{\mathbf{t}})\mathbf{N_t}, \tag{24}$$

which leads to $\sigma(s) = \frac{\tau_{\mathbf{t}}}{\kappa_{\mathbf{t}}} = \cot[\phi]$.

($\Leftarrow$) Suppose $\sigma = \cot[\phi]$, i.e., $\tau_{\mathbf{t}} = \cot[\phi]\kappa_{\mathbf{t}}$ and let us consider the vector

$$\mathbf{d} = \cos[\phi]\mathbf{T_t} + \sin[\phi]\mathbf{B_t}. \tag{25}$$

From the Frenet formula (2), it is easy to prove the vector $\mathbf{d}$ is constant and $\langle \mathbf{T_t}, \mathbf{d} \rangle = \cos[\phi]$. This concludes the proof of lemma (4.4).

Lemma 4.5. *Let $\psi : I \to \boldsymbol{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. The curve ψ is a 1-slant helix or slant helix if and only the unit Darboux (modified Darboux [11]) vector field $\boldsymbol{B}_{\mathbf{t}} = \frac{f\boldsymbol{T}+\boldsymbol{B}}{\sqrt{1+f^2}}$ of ψ makes a constant angle with fixed direction.*

Proof: ($\Rightarrow$) The proof of the necessary condition is the same as the necessary condition of the above lemma.

($\Leftarrow$) Let $\mathbf{d}$ be the unitary fixed vector makes a constant angle, $\frac{\pi}{2} - \phi$, with the vector $\mathbf{B_t} = \frac{f\mathbf{T}+\mathbf{B}}{\sqrt{1+f^2}}$. Therefore

$$\langle \mathbf{B_t}, \mathbf{d} \rangle = \sin[\phi]. \tag{26}$$

Differentiating the equation (26) with respect to the variable $s_{\mathbf{t}}$ and using Frenet equations (2), we get

$$-\tau_{\mathbf{t}}\langle \mathbf{N_t}, \mathbf{d} \rangle = 0. \tag{27}$$

Because $\tau_{\mathbf{t}} = \sigma\sqrt{1+f^2} \neq 0$, then we have

$$\langle \mathbf{N_t}, \mathbf{d} \rangle = 0. \tag{28}$$

From the above equation, the vector $\mathbf{d}$ is perpendicular to the vector $\mathbf{N_t}$ and so that the vector $\mathbf{d}$ lies in the space consists with the vectors $\mathbf{B_t}$ and $\mathbf{T_t}$. Therefore the vector $\mathbf{d}$

makes a constant angles with the two vectors $\mathbf{B_t}$ and $\mathbf{T_t}$. This concludes the proof of lemma (4.5).

(3): The 2-*slant helix* is the curve whose the unit vector

$$\psi_3 = \frac{\psi_2'(s)}{\|\psi_2'(s)\|} = \frac{\mathbf{N}'(s)}{\|\mathbf{N}'(s)\|} = \frac{-\mathbf{T} + f\mathbf{N}}{\sqrt{1+f^2}}, \tag{29}$$

makes a constant angle with a fixed direction. So that the 2-slant helix is a new special curves we can call it *slant-slant helix*.

If we using the Frenet frame (9) of the principal normal indicatrix of the the curve ψ, it is easy to prove the following two new lemmas.

Lemma 4.6. *Let $\psi : I \to \boldsymbol{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. The curve ψ is a 2-slant helix or slant-slant helix (the vector ψ_3 makes a constant angle, ϕ, with a fixed straight line in the space) if and only if the function $\Gamma(s) = \frac{\tau_{\mathbf{n}}}{\kappa_{\mathbf{n}}} = \cot[\phi]$.*

The proof of the above lemma (using the Frenet frame (9)) is similar as the proof of lemma (4.4) (using the Frenet frame (2)).

Lemma 4.7. *Let $\psi : I \to \boldsymbol{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. The curve ψ is a 2-slant helix or slant-slant helix if and only if the vector $\boldsymbol{B}_{\mathbf{n}} = \frac{1}{\sqrt{1+\sigma^2}}\Big[\frac{f\,\boldsymbol{T}+\boldsymbol{B}}{\sqrt{1+f^2}} + \sigma\,\boldsymbol{N}\Big]$ makes a constant angle with fixed direction.*

The proof of the above lemma (using the Frenet frame (9)) is similar as the proof of lemma (4.5) (using the Frenet frame (2)).

(4): The 3-*slant helix* is the curve whose the unit vector

$$\psi_4 = \frac{\psi_3'(s)}{\|\psi_3'(s)\|} = \frac{\sigma}{\sqrt{1+\sigma^2}}\Big[\frac{f\,\mathbf{T}+\mathbf{B}}{\sqrt{1+f^2}} - \frac{\mathbf{N}}{\sigma}\Big], \tag{30}$$

makes a constant angle with a fixed direction. So that the 2-slant helix is a new special curves we can call it *slant-slant-slant helix*.

Lemma 4.8. *Let $\psi : I \to \boldsymbol{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. The curve ψ is a 3-slant helix or slant-slant-slant helix (the vector ψ_4 makes a constant angle, ϕ, with a fixed straight line in the space) if and only if the function*

$$\Lambda = \frac{\Gamma'(s)}{\kappa(s)\sqrt{1+f^2(s)}\sqrt{1+\sigma^2(s)}\Big(1+\Gamma^2(s)\Big)^{3/2}} = \cot[\phi]. \tag{31}$$

proof: ($\Rightarrow$) Let $\mathbf{d}$ be the unitary fixed vector makes a constant angle, ϕ, with the vector $\psi_4 = \mathbf{N_n}$. Therefore

$$\langle \mathbf{N_n}, \mathbf{d} \rangle = \cos[\phi]. \tag{32}$$

Differentiating the equation (32) with respect to the variable $s_{\mathbf{n}} = \int \kappa(s)\sqrt{1+f^2(s)}ds$ and using the Frenet equations (9), we get

$$\langle -\kappa_{\mathbf{n}} \mathbf{T_n} + \tau_{\mathbf{n}} \mathbf{B_n}, \mathbf{d} \rangle = 0. \tag{33}$$

Therefore,

$$\langle \mathbf{T_n}, \mathbf{d} \rangle = \frac{\tau_{\mathbf{n}}}{\kappa_n} \langle \mathbf{B_n}, \mathbf{d} \rangle = \Gamma \langle \mathbf{B_n}, \mathbf{d} \rangle.$$

If we put $\langle \mathbf{B_n}, \mathbf{d} \rangle = g(s)$, we can write

$$\mathbf{d} = \Gamma\, g\, \mathbf{T_n} + \cos[\phi] \mathbf{N_n} + g\, \mathbf{B}_n.$$

From the unitary of the vector $\mathbf{d}$ we get $g = \pm \frac{\sin[\phi]}{\sqrt{1+\Gamma^2}}$. Therefore, the vector $\mathbf{d}$ can be written as

$$\mathbf{d} = \pm \frac{\Gamma\, \sin[\phi]}{\sqrt{1+\Gamma^2}}\, \mathbf{T_n} + \cos[\phi]\, \mathbf{N_n} \pm \frac{\sin[\phi]}{\sqrt{1+\Gamma^2}}\, \mathbf{B_n}. \tag{34}$$

The equation (33) can be written in the form:

$$\langle -\mathbf{T_n} + \Gamma\, \mathbf{B_n}, \mathbf{d} \rangle = 0. \tag{35}$$

If we differentiate the equation (33) with respect to $s_{\mathbf{n}}$, again, we obtain

$$\langle \dot{\Gamma}\, \mathbf{B_n} + (1+\Gamma^2)\sqrt{1+\sigma^2} \mathbf{N_n}, \mathbf{d} \rangle = 0, \tag{36}$$

where dot is the differentiation with respect to $s_{\mathbf{n}}$. If we put the vector $\mathbf{d}$ from equation (34) in the equation (36), we obtain the following condition

$$\frac{\dot{\Gamma}}{\sqrt{1+\sigma^2}(1+\Gamma^2)^{3/2}} = \pm\, \cot[\phi].$$

Finally, $s_{\mathbf{n}} = \int \kappa(s)\sqrt{1+f^2(s)}ds$ and $\dot{\Gamma} = \frac{\Gamma'(s)}{\kappa(s)\sqrt{1+f^2(s)}}$, we express the desired result.

($\Leftarrow$) Suppose that $\frac{\dot{\Gamma}}{\sqrt{1+\sigma^2}(1+\Gamma^2)^{3/2}} = \pm\, \cot[\phi]$ where . is the differentiation with respect to $s_{\mathbf{n}}$. Let us consider the vector

$$\mathbf{d} = \pm\, \cos[\phi] \Big(\frac{\Gamma\, \tan[\phi]}{\sqrt{1+\Gamma^2}}\, \mathbf{T_n} \pm \mathbf{N_n} + \frac{\tan[\phi]}{\sqrt{1+\Gamma^2}}\, \mathbf{B_n} \Big).$$

We will prove that the vector $\mathbf{d}$ is a constant vector. Indeed, applying Frenet formula (9)

$$\dot{\mathbf{d}} = \pm\sqrt{1+\sigma^2}\, \cos[\phi] \Big(\pm\, \mathbf{T_n} + \frac{\Gamma \tan[\phi]}{\sqrt{1+\Gamma^2}}\, \mathbf{N}_n \mp \mathbf{T_n} \pm \Gamma \mathbf{B} \mp \Gamma \mathbf{B_n} - \frac{\Gamma \tan[\phi]}{\sqrt{1+\Gamma^2}}\, \mathbf{N}_n \Big) = 0$$

Therefore, the vector $\mathbf{d}$ is constant and $\langle \mathbf{N_n}, \mathbf{d}\rangle = \cos[\phi]$. This concludes the proof of lemma (4.8).

From the section (3), we can see that:

(i): The function $f(s)$ is equal the ratio of the torsion ($\tau = \tau_0$) and curvature ($\kappa = \kappa_0$) of the curve $\psi = \psi_0$ and may be named it $\sigma_0(s) = f(s) = \frac{\tau_0(s)}{\kappa_0(s)}$.

(ii): The function $\sigma(s)$ is equal the ratio of the torsion ($\tau_{\mathbf{t}} = \tau_1$) and curvature ($\kappa_{\mathbf{t}} = \kappa_1$) of the tangent indicatrix $\mathbf{T} = \psi_1$ of the curve ψ and may be named it $\sigma_1(s) = \sigma(s) = \frac{\tau_1(s)}{\kappa_1(s)}$.

(iii): The function $\Gamma(s)$ is equal the ratio of the torsion ($\tau_{\mathbf{n}} = \tau_2$) and curvature ($\kappa_{\mathbf{n}} = \kappa_2$) of the principal normal indicatrix $\mathbf{N} = \psi_2$ of the curve ψ and may be named it $\sigma_2(s) = \Gamma(s) = \frac{\tau_2(s)}{\kappa_2(s)}$.

We expect that: the function $\Lambda(s)$ is equal the ratio of the torsion τ_3 and curvature κ_3 of the spherical image of ψ_3 indicatrix and may be named it $\sigma_3(s) = \Lambda(s) = \frac{\tau_3(s)}{\kappa_3(s)}$. So that, we can write (the proof is classical) the following lemma:

Lemma 4.9. *If the Frenet frame of the spherical image of $\psi_3 = \frac{-\boldsymbol{T}+f\boldsymbol{B}}{\sqrt{1+f^2}}$ indicatrix of the curve ψ is $\{\boldsymbol{T}_3, \boldsymbol{N}_3, \boldsymbol{B}_3\}$, then we have Frenet formula:*

$$\begin{bmatrix} \boldsymbol{T}_3'(s_3) \\ \boldsymbol{N}_3'(s_3) \\ \boldsymbol{B}_3'(s_3) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_3 & 0 \\ -\kappa_3 & 0 & \tau_3 \\ 0 & -\tau_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_3(s_3) \\ \boldsymbol{N}_3(s_3) \\ \boldsymbol{B}_3(s_3) \end{bmatrix}, \tag{37}$$

where

$$\begin{cases} \boldsymbol{T}_3 = \frac{\sigma}{\sqrt{1+\sigma^2}}\Big[\frac{f\,\boldsymbol{T}+\boldsymbol{B}}{\sqrt{1+f^2}} - \frac{\boldsymbol{N}}{\sigma}\Big], \\ \boldsymbol{N}_3 = \frac{1}{\sqrt{1+\sigma^2}\sqrt{1+\Gamma^2}}\Big[\frac{\Gamma\big(f\,\boldsymbol{T}+\boldsymbol{B}\big)+\sqrt{1+\sigma^2}\big(\boldsymbol{T}-f\boldsymbol{B}\big)}{\sqrt{1+f^2}} + \sigma\Gamma\boldsymbol{N}\Big], \\ \boldsymbol{B}_3 = \frac{1}{\sqrt{1+\sigma^2}\sqrt{1+\Gamma^2}}\Big[\frac{f\,\boldsymbol{T}+\boldsymbol{B}-\Gamma\sqrt{1+\sigma^2}\big(\boldsymbol{T}-f\boldsymbol{B}\big)}{\sqrt{1+f^2}} + \sigma\boldsymbol{N}\Big], \end{cases} \tag{38}$$

and

$$s_3 = \int \kappa(s)\sqrt{1+f^2(s)}\sqrt{1+\sigma^2(s)}\,ds, \quad \kappa_3 = \sqrt{1+\Gamma^2}, \quad \tau_3 = \Lambda\sqrt{1+\Gamma^2}, \tag{39}$$

where s_3 is the natural representation of the spherical image of ψ_3 indicatrix of the curve ψ and κ_3 and τ_3 are the curvature and torsion of this curve.

Therefore it is easy to see that:

$$\frac{\tau_3}{\kappa_3} = \Lambda = \sigma_3. \tag{40}$$

If we using the Frenet frame (38) of the spherical image of ψ_3 indicatrix of the curve ψ, it is easy to prove the following new lemma.

Lemma 4.10. *Let $\psi : I \to \boldsymbol{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. The curve ψ is a 3-slant helix or slant-slant-slant helix if and only if the vector $\boldsymbol{B}_3 = \frac{1}{\sqrt{1+\sigma^2}\sqrt{1+\Gamma^2}}\Big[\frac{f\,\boldsymbol{T}+\boldsymbol{B}-\Gamma\sqrt{1+\sigma^2}\big(\boldsymbol{T}-f\,\boldsymbol{B}\big)}{\sqrt{1+f^2}} + \sigma\boldsymbol{N}\Big]$ makes a constant angle with fixed direction.*

The proof of the above lemma (using the Frenet frame (38)) is similar as the proof of lemma (4.5) (using the Frenet frame (2)).

5 General results

From the above discussions, we can introduce an important lemmas for the k-slant helix in general form as follows:

Lemma 5.1. *If the Frenet frame of the spherical image of ψ_k = indicatrix of the curve ψ is $\{\boldsymbol{T}_k, \boldsymbol{N}_k, \boldsymbol{B}_k\}$, then we have Frenet formula:*

$$\begin{bmatrix} \boldsymbol{T}_k'(s_k) \\ \boldsymbol{N}_k'(s_k) \\ \boldsymbol{B}_k'(s_k) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_k & 0 \\ -\kappa_k & 0 & \tau_k \\ 0 & -\tau_k & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_k(s_k) \\ \boldsymbol{N}_k(s_k) \\ \boldsymbol{B}_k(s_k) \end{bmatrix}, \tag{41}$$

where

$$\boldsymbol{T}_k = \psi_{k+1}, \quad \boldsymbol{N}_k = \psi_{k+2}, \quad \boldsymbol{B}_k = \frac{\psi_{k+1} \times \psi_{k+2}}{\|\psi_{k+1} \times \psi_{k+2}\|}, \tag{42}$$

and

$$\begin{cases} s_k = \int \kappa(s)\sqrt{1+\sigma_0^2(s)}\sqrt{1+\sigma_1^2(s)}\ldots\sqrt{1+\sigma_{k-1}^2(s)}\,ds, \\ \kappa_k = \sqrt{1+\sigma_{k-1}^2}, \\ \tau_k = \sigma_k\sqrt{1+\sigma_{k-1}^2}, \end{cases} \tag{43}$$

where

$$\sigma_k = \frac{\sigma_{k-1}'}{\kappa(s)\sqrt{1+\sigma_0^2(s)}\sqrt{1+\sigma_1^2(s)}\ldots\Big(1+\sigma_{k-1}^2(s)\Big)^{3/2}}, \tag{44}$$

s_k is the natural representation of the spherical image of ψ_k indicatrix of the curve ψ and κ_k and τ_k are the curvature and torsion of this curve.

From the the above lemma we have $\frac{\tau_k}{\kappa_k} = \sigma_k$, which leads the following lemma:

Lemma 5.2. *Let $\psi : I \to \boldsymbol{E}^3$ be a k-slant helix. The spherical image of ψ_k indicatrix of the curve ψ is a spherical helix.*

Lemma 5.3. *Let $\psi : I \to \boldsymbol{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. The curve ψ is a k-slant helix (the vector ψ_{k+1} makes a constant angle, ϕ, with a fixed straight line in the space) if and only if the function*

$$\sigma_k = \cot[\phi]. \tag{45}$$

Lemma 5.4. *Let $\psi : I \to \boldsymbol{E}^3$ be a curve that is parameterized by arclength with intrinsic equations $\kappa(s) \neq 0$ and $\tau(s) \neq 0$. The curve ψ is a k-slant helix if and only if the vector $\boldsymbol{B}_k = \frac{\psi_{k+1} \times \psi_{k+2}}{\|\psi_{k+1} \times \psi_{k+2}\|}$ makes a constant angle with fixed direction.*

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